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1999 J. Phys. A: Math. Gen. 32 5179

(http://iopscience.iop.org/0305-4470/32/28/301)

J. Phys. A: Math. Gen. **32** (1999) 5179–5199. Printed in the UK PII: S0305-4470(99)00317-0

Sailing the deep blue sea of decaying Burgers turbulence*

Michel Bauer and Denis Bernard†

Service de Physique Théorique de Saclay‡, F-91191, Gif-sur-Yvette, France

E-mail: bauer@spht.saclay.cea.fr and dbernard@spht.saclay.cea.fr

Received 22 December 1998, in final form 10 May 1999

Abstract. We study Lagrangian trajectories and scalar transport statistics in decaying Burgers turbulence. We choose velocity fields solutions of the inviscid Burgers equation whose probability distributions are specified by Kida's statistics. They are time-correlated, and neither time-reversal invariant nor Gaussian. We discuss in some detail the effect of shocks on trajectories and transport equations. We derive the inviscid limit of these equations using a formalism of operators localized on shocks. We compute the probability distribution functions of the trajectories although they do not define Markov processes. As physically expected, these trajectories are statistically well defined but collapse with probability one at infinite time. We point out that the advected scalars enjoy inverse energy cascades. We also make a few comments on the connection between our computations and persistence problems.

1. Introduction

Lagrangian trajectories driven by a velocity field $u(x,t)$ are solutions of the differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(x(t),t). \tag{1}$$

As known from the works of Richardson, Kolmogorov and Batchelor, for example, [1], they acquire peculiar properties when the flow becomes turbulent. These properties are probably going to play an important role in the understanding of fully developed turbulence. For example, the recent proof [6] of the existence and uniqueness of the stationary state for the inviscid forced Burgers turbulence is based on an analysis of these trajectories.

Statistical properties of these trajectories may be deciphered by looking at transport phenomena in turbulent systems. Recent studies of Kraichnan's advection models [2] have made these expected properties more explicit. Kraichnan's models assume that the velocity fields are Gaussian and white-noise in time. These simplifications lead to the solvability of the models. See [3] for recent studies of the Kraichnan models for incompressible fluids and [4, 5] for compressible ones. Two kinds of behaviour have been observed:

(1) Statistical ill-definedness, meaning that two trajectories starting at the same point have a non-vanishing probability to be far apart at later time.

* Or 'Lagrangian trajectories in decaying Burgers turbulence'.
† Membre du CNRS.
‡ Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique.

(2) Trajectory collapse for compressible enough fluids, meaning that two trajectories starting initially at different positions have a non-zero probability to follow the same path after some time.
However:
(3) Properties (1) and (2) do not seem to occur simultaneously.

The motivation of the present work is to decipher whether these properties are more robust and hold true for more realistic velocity fields than those chosen in Kraichnan's models. Of course we could not solve the problem with a velocity field describing a real three-dimensional turbulent system. Instead we shall consider (unrealistic) velocity fields, solutions of the Burgers equation which in 1 + 1 dimensions takes the form:

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = 0 \tag{2}$$

where $u = u(x, t)$ is the (compressible) velocity field and ν the viscosity. This is a variant of the Navier–Stokes equation in which the role of the pressure has been neglected. Although we shall stick to one-dimensional space, some of the following considerations could be generalized to higher dimensions.

No external force is applied to equation (2). So its inviscid limit $\nu \to 0$ corresponds to decaying turbulence whose statistical description consists in finding a probability distribution of the velocity fields solution of equation (2) given random initial data. One usually expects a more universal behaviour at large time. Thus, we shall consider a family of velocity fields, solutions of the inviscid limit $\nu \to 0$ of the Burgers equation, whose probability distribution describes the long-time behaviour of large classes of initial conditions. These velocity statistics are those first introduced by Kida [8]. In contrast to the Kraichnan model, the velocity fields are then not white-noise in time, and neither time-reversal invariant nor Gaussian.

For compressible fluids, one may look at two kinds of transport phenomena depending on whether one is looking at the advection of a tracer, that we shall denote by $T(x, t)$, or at the advection of the density of a pollutant, that we shall denote by $\rho(x, t)$. The corresponding viscosity is written κ. The equations governing these transports are

$$\partial_t T(x, t) + u(x, t)\partial_x T(x, t) - \kappa\partial_x^2 T(x, t) = 0 \tag{3}$$

$$\partial_t \rho(x, t) + \partial_x (u(x, t)\rho(x, t)) - \kappa\partial_x^2 \rho(x, t) = 0. \tag{4}$$

They differ by the order of the derivative and velocity.

In the inviscid limit $\nu \to 0$, solutions of the Burgers equation develop shocks at which the velocity is not smooth. This non-smoothness implies that the naive definition of the trajectories does not apply. Therefore, these trajectories and the transport equations have to be dealt with carefully. As we shall see, a correct definition of the transport equations will turn out to be

$$\partial_t T(x, t) + \tfrac{1}{2}(u(x^+, t) + u(x^-, t))\partial_x T(x, t) - \kappa\partial_x^2 T(x, t) = 0 \tag{5}$$

$$\partial_t \rho(x, t) + \partial_x \tfrac{1}{2}((u(x^+, t) + u(x^-, t))\rho(x, t)) - \kappa\partial_x^2 \rho(x, t) = 0 \tag{6}$$

with $u(x^\pm, t) = \lim_{\epsilon\to 0^+} u(x \pm \epsilon, t)$. Although equations (5), (6) seem to be naively equivalent to equations (3), (4), they are not since in the inviscid limit the velocity field $u(x, t)$ is not smooth.

In the limit $\kappa \to 0$, equations (5), (6) have a natural interpretation in terms of Lagrangian trajectories. However, the naive equation (1), which is actually meaningless since $u(x, t)$ is discontinuous, has to be modified into

$$\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_+ = \frac{1}{2}(u(x(t)^+, t) + u(x(t)^-, t)). \tag{7}$$

Again this differs from equation (1) because $u(x,t)$ is not smooth. The physical meaning of this modification is clear. At points of discontinuity, the ill-defined velocity is replaced by the velocity of the shock which, as has been well known for some time [7–9], is just the average of the velocities just before and just after the shock. Once this has been performed, we shall describe how to compute the probability distribution functions (PDF) of the trajectories and we shall use them to discuss the properties of the transport equation (5) in the limit $\kappa \to 0$.

This paper is organized as follows. In the following section we recall basic facts concerning the Burgers equation and the velocity profiles we shall use. Section 3 is devoted to giving a precise definition of Lagrangian trajectories in the inviscid limit $\nu \to 0$ and to the relation with the correct form of transport equations and their solutions. In section 4 we establish identities, called equations of motion, which are valid inside correlation functions. This is based on operators localized at shocks and their algebra. In section 5, the backward and forward probability distribution functions of the trajectories are introduced and their formal properties emphasized. We use the identities established in section 4 to verify that these PDFs are solutions of the transport equations. In section 6, we make explicit computations for one- and two-particle distributions. We check the consistency with the expected physical properties of the trajectories. In particular, we show by different approaches that the trajectories are statistically well-defined but that particles have a non-vanishing probability to collapse. This is in agreement with the general properties of Lagrangian trajectories mentioned above as (1)–(3). As we deal with a highly compressible fluid, the alternative (2) is realized and the alternative (1) is excluded. The connexion with persistence problems is made. Finally, arguments indicating that the energy cascade in scalar advection in these flows is inverse, i.e. towards the large scale, are presented in section 7.

2. Velocity profiles

This short section is devoted to the specification of the statistics of the velocity profiles to be used in this paper.

• In order to fix notations, we recall a few elementary facts concerning the Burgers equation (see e.g. [7–9] and references therein). As is well known, the equation is solved by implementing the Cole–Hopf transformation which maps it to the heat equation. This works as follows. Let $Z(x,t) = \exp[-\frac{1}{2\nu}\Phi(x,t)]$ where $u(x,t) = \partial_x \Phi(x,t)$. Equation (2) for u is mapped into the heat equation for Z:

$$[\partial_t - \nu\partial_x^2]Z(x,t) = 0.$$

Thus, given the initial condition $u(x, t=0) \equiv u_0(x)$, the velocity field at a later time t is recovered from the potential $\Phi(x,t)$ given by the relation

$$\exp\left[-\frac{1}{2\nu}\Phi(x,t)\right] = \int \frac{\mathrm{d}y}{\sqrt{4\pi\nu t}} \exp\left[-\frac{1}{2\nu}\left(\Phi_0(y) + \frac{(x-y)^2}{2t}\right)\right] \tag{8}$$

with $\Phi_0(x)$ standing for the initial potential such that $u_0(x) = \partial_x\Phi_0(x)$. The inviscid Burgers equation corresponds to the limit $\nu \to 0$. The solution is then given by solving a minimalization problem

$$u(x,t) = \partial_x\Phi(x,t) \qquad \text{with} \quad \Phi(x,t) = \min_y \left(\Phi_0(y) + \frac{(x-y)^2}{2t}\right). \tag{9}$$

Outside shocks the minimum is reached for only one value y_* of y, the solution of the equation $u_0(y_*)t = x - y_*$. The velocity is $u(x,t) = \frac{x-y_*}{t} = u_0(y_*)$. It is effectively a local solution of the inviscid Burgers equation since, by the minimum condition defining y_*, we have $u(x,t) = u_0(x - tu(x,t))$. A simple geometrical construction of the solution (9) is described in [7, 8]. For large t, y_* coincides approximately with one of the local minima of $\Phi_0(y)$ and it practically does not change under small variations of x so that, in between the shocks, the velocity is approximately linear with the slope $\frac{1}{t}$.

Shocks appear when the minimum is reached for two values y_1 and y_2 of y. Let $\Phi_{1,2} = \Phi_0(y_{1,2})$ be the value of the initial potential at these points. Then equation (8) allows one to determine the velocity profile $u_s(x,t)$ around and inside the shocks at finite value of the viscosity ν by expressing $\exp[-\frac{1}{2\nu}\Phi_s(x,t)]$ as the sum of contributions from the two minima. One obtains

$$u_s(x,t) = \frac{1}{t}\left(x - \frac{1}{2}(y_1 + y_2)\right) - \frac{\mu_s}{2t}\tanh\left(\frac{\mu_s}{4\nu t}\left(x - \xi_s t - \frac{1}{2}(y_1 + y_2)\right)\right) \tag{10}$$

where $\mu_s = y_1 - y_2 > 0$ and $\xi_s = \frac{\Phi_1 - \Phi_2}{y_1 - y_2}$. The width of the shock is of order $l_c \simeq \frac{2\nu t}{\mu_s}$. In the inviscid limit $\nu \to 0$, equation (10) becomes

$$u_s(x,t)|_{\nu=0} = \xi_s \mp \frac{\mu_s}{2t} + \frac{x - x_s(t)}{t} \qquad \text{for} \quad \pm(x - x_s(t)) > 0 \tag{11}$$

where $x_s(t) = \xi_s t + \frac{1}{2}(y_1 + y_2)$ is the time t position of the shock which moves with the velocity ξ_s and follows a Lagrangian trajectory. The values of the velocity on the two sides of the shock are

$$u_s^\pm \equiv u_s(x_s^\pm) = \xi_s \mp \frac{\mu_s}{2t} \tag{12}$$

so that $\frac{\mu_s}{t}$ is the amplitude of the shock.

• To mimic this large-time behaviour, following Kida [8], we choose as velocity profiles the ansatz

$$u(x,t) = \partial_x S(x,t) \qquad \text{with} \quad S(x,t) = \min_j\left(\phi_j + \frac{(x - y_j)^2}{2t}\right). \tag{13}$$

The points $(\phi_j, y_j)_{j\in\mathbb{Z}}$ specify a given realization. For any realization, i.e. for any data of the points (ϕ_j, y_j), these ansätze (13) are solutions of the inviscid Burgers equation. They have exact sawtooth profiles† with slope $1/t$ (see figure 1). In this ansatz all shocks are created at time $t = 0$. The later time evolution is then governed by the shock collisions. Thus different times are strongly correlated.

Following Kida [8], we shall concentrate on the velocity statistics specified by demanding that $(\phi_j, y_j)_{j\in\mathbb{Z}}$ be a Poisson point process‡ with intensity $\mathcal{J} = \mathrm{e}^{\phi}\,\mathrm{d}\phi\,\mathrm{d}y$.

This choice of statistics ensures that the velocity $u(x,t)$ is self-similar with characteristic length $l(t) \sim \sqrt{t}$ which means that $su(sx, s^2t) \cong u(x,t)$. Here and in the following, $\cong$ means an equality in law, i.e. inside any correlation functions. We could as well choose other intensities for the Poisson process. This amounts to choosing other scalings for the characteristic length.

3. Lagrangian trajectories and transport equations

Lagrangian trajectories $x(t)$ starting at point x_0 at time t_0 are defined as solutions of the evolution equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(x(t),t) \qquad \text{with} \quad x(t_0) = x_0. \tag{14}$$

† In particular, the velocity is not defined by the above formulae at the shocks. This is at the origin of most of the following discussions.

‡ The basic rules to manipulate such processes are briefly recalled in appendices A and B, where some explicit computations are made.

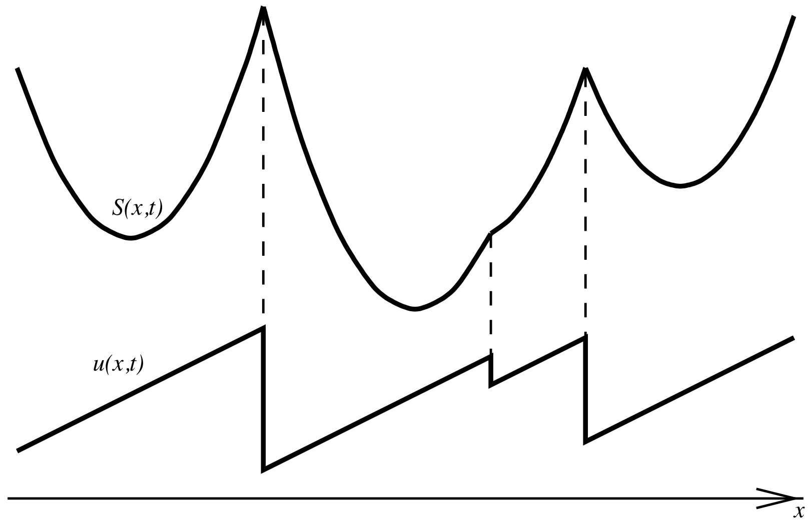

Figure 1. The sawtooth velocity profile.

In this section, we shall specify the equation governing Lagrangian trajectories in the inviscid case. This first requires a detour through the $\nu \neq 0$ situation.

• The above differential equation is well-posed for a velocity field $u(x,t)$ solution of the Burgers equation with finite non-vanishing viscosity $\nu \neq 0$, since then $u(x,t)$ is smooth enough.

However, the limit $\nu \to 0$ is delicate:

(i) If the point $x(t)$ of a trajectory is far from shocks, the velocity is then regular around that point even in the inviscid limit and the trajectory is well defined. At large time, the velocity far from shocks is of the form $u(x,t) = \frac{1}{t}(x - y_*)$ with y_* approximately constant and the trajectories are then straight lines. This applies as long as the trajectories are away from shocks.

(ii) Assuming that shocks are diluted, the trajectories near a shock in the inviscid limit $\nu \to 0$ may be analysed using the velocity profile (11). In this environment, solutions of the Lagrange equation $\dot{x} = u(x,t)$ are such that:

$$\sinh\left(\frac{\mu_s}{4\nu t}(x(t) - x_s(t))\right)\exp\left(-\frac{\mu_s^2}{8\nu t}\right) = \text{constant}$$

with $x_s(t)$ the time t position of the centre of the shock. Recall that the width of the shock is of order $l_c \simeq \frac{2\nu t}{\mu_s}$. This equation means that particles away from the shock take a finite time to enter the shock. Once they are in the shock they move coherently with it with velocity almost equal to $\xi_s \equiv \dot{x}_s(t)$. But they never cross the shock centre.

• If we want to recover the $\nu = 0$ limit behaviour directly in the inviscid case with the ansatz (13) for the velocities, we have to be careful. At discontinuities of the velocity, equation (14) does not make sense for two reasons: the velocity is not defined at the shocks and the derivative of a differentiable function cannot exhibit pure discontinuities.

A simple modification that will ensure the gluing of particles to shocks, the main feature at finite but small viscosity, is the following:

(i) First we define $\overline{u}(x,t) \equiv \frac{1}{2}(u(x^+,t)+u(x^-,t))$. For the ansatz (13), this definition makes sense for any x and extends the definition of $u(x,t)$ to shocks (obviously $u = \overline{u}$ away from shocks).

(ii) Then we demand that trajectories be continuous and satisfy

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}|_+ \equiv \lim_{\epsilon\to 0^+} \frac{x(t+\epsilon)-x(t)}{\epsilon} = \overline{u}(x,t). \tag{15}$$

If we assume that the shocks form a discrete set (no limit points)† these two requirements ensure that trajectories are uniquely defined for $t \geqslant t_0$ once the boundary condition $x(t_0) = x_0$ is specified. Since the velocity of a shock is the mean of the velocities at the points just preceding and just following it, equation (15) ensures that particles stick to shocks.

• According to the ansatz (13), away from shocks, $u(x,t) = \frac{1}{t}(x-y)$ for some y. So the trajectory is

$$x(t) = x_0 + (t-t_0)\frac{x_0 - y}{t_0}, \qquad \text{away from shocks} \tag{16}$$

with x_0 the position at time t_0. This is true up to the time at which the particle meets a shock. Shocks are at the points where two parabolae $\phi_{1,2} + \frac{(x-y_{1,2})^2}{2t}$ minimizing equation (13) intersect. They move with a velocity $\xi_{12} = \frac{\phi_1-\phi_2}{y_1-y_2}$. In the time interval during which the shock exists, the trajectory equation is the shock equation:

$$x(t) = \tfrac{1}{2}(y_1+y_2) + \xi_{12}t \qquad \text{on the shock.} \tag{17}$$

Once a particle is on a shock it follows it and the cascade of shocks arising from its collisions. Note that when two shocks hit they merge into a third shock. In particular, a particle not on a shock at time t has never met a shock before‡. A general feature of the trajectories is that particles move at constant velocity on intervals of the form $[t,t'[$ with $t < t'$.

• This definition of the Lagrangian trajectories ensures the physical fact that the velocity field in the inviscid limit is transported by the fluid. Indeed, since a particle moving along a Lagrangian trajectory keeps its velocity for a finite time interval ϵ with ϵ sufficiently small, one has

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}[\overline{u}(x+\epsilon\overline{u}(x,t),t+\epsilon) - \overline{u}(x,t)] = 0.$$

Accordingly, the transport equation for a tracer $T(x,t)$ moving in the inviscid velocity field $u(x,t)$ will be

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}[T(x+\epsilon\overline{u}(x,t),t+\epsilon) - T(x,t)] = 0. \tag{18}$$

It coincides with the $\kappa \to 0$ limit of equation (5) provided the Lagrangian trajectories are specified as in (15).

We show now how the above equation (18) can be solved. The idea is to find an implicit formula for the Lagrangian trajectories, taking any number of shocks into account. Fix x_0 and t_0, and consider the function $\mathcal{X}(x,t) \equiv x - x_0 - (t-t_0)\overline{u}(x,t)$ for $t \geqslant t_0$ and x arbitrary. It is readily checked that $\frac{\mathcal{X}(x,t)-\mathcal{X}(x',t)}{x-x'} \geqslant \frac{t_0}{t}$, so that for fixed t, $\mathcal{X}(x,t)$ is a strictly increasing function of x with $\lim_{x\to\pm\infty} \mathcal{X}(x,t) = \pm\infty$. This means that we can define a function $\tilde{x}(t)$ for $t \geqslant t_0$ by the condition that $\mathcal{X}(\tilde{x}(t)^+,t) \geqslant 0 \geqslant \mathcal{X}(\tilde{x}(t)^-,t)$. It is cumbersome but

† The probability distribution for the velocities ensures that this happens with probability one.

‡ A different proof of the same result can be found in appendix B where it appears as a natural part of the argument.

straightforward to check that $\tilde{x}(t)$ is the solution of (15) with initial condition $\tilde{x}(t_0) = x_0$. Hence the solution of (18) with initial condition $T(x, t_0) = \theta(x - x_0)$, where $\theta(x)$ is the Heaviside step function, is $T(x, t) = \theta(x - x_0 - (t - t_0)\overline{u}(x, t))$. By linearity, the solution with initial data $T(x, t_0) = T_0(x)$ is $T(x, t) = T_0(x - (t - t_0)\overline{u}(x, t))$. This solution develops discontinuities† at the shocks, even if the initial condition is smooth.

4. Operator localized on shocks and equations of motion

In this section, we discuss what happens to the Burgers equation in the inviscid limit. We shall argue that the actual inviscid Burgers equation is not the naive limit $\nu \to 0$ of equation (2) but is

$$[\partial_t u(x, t) + \tfrac{1}{2}(u(x^+, t) + u(x^-, t))(\partial_x u(x, t))] = 0 \tag{19}$$

with the equality valid inside correlation functions with velocity fields (with or without derivatives) away from x and velocity fields (without derivatives) at the point x. This is not quite the usual way to write the inviscid Burgers equation. So we shall start with the more familiar formulae and show the equivalence with (19). The argument will be based on an analysis of operators localized on shocks which may be used to derive equations of motion valid inside any correlation functions.

• At $\nu \neq 0$ the Burgers equation (2) could be written as

$$(\partial_t + u\partial_x - \nu\partial_x^2 + \lambda^2\nu(\partial_x u)^2)\mathrm{e}^{\lambda u} = 0. \tag{20}$$

Since $\mathrm{e}^{\lambda u}$ is finite in the inviscid regime, the distribution $\partial_x^2\mathrm{e}^{\lambda u}$ is well defined in this limit, and $\nu\partial_x^2\mathrm{e}^{\lambda u}$ vanishes when $\nu \to 0$. Equation (20) can be rewritten in this limit as

$$(\partial_t u(x, t) + u(x, t)\partial_x u(x, t))\mathrm{e}^{\lambda u(x,t)} + \lambda\epsilon(x, t)\mathrm{e}^{\lambda u(x,t)} \cong 0. \tag{21}$$

Here $\epsilon(x, t)$ is the dissipation field defined by $\epsilon(x, t) = \lim_{\nu\to 0} \nu(\partial_x u)^2$. The product $u(x, t)\partial_x u(x, t)$ is ill-defined since it is a product of distributions. Equation (21) should actually be read as

$$\left(\partial_t + \lambda\partial_\lambda \frac{1}{\lambda}\partial_x\right)\mathrm{e}^{\lambda u(x,t)} + \lambda^2\epsilon(x, t)\mathrm{e}^{\lambda u(x,t)} \cong 0. \tag{22}$$

This is the well known inviscid equation of motion. The fact that the dissipation field survives the inviscid limit is sometimes called the dissipative anomaly.

• The presence of shocks is at the origin of universal features which are independent of the details of the statistics. As explained in [10], they may be analysed by looking at fields localized on the shocks. By definition, these fields may be represented for any realization as

$$\mathcal{O}_g(x, t) = \sum_{\text{shocks}} g(\xi_s, \mu_s)\delta(x - x_s(t)) \tag{23}$$

where the sum is over the shocks with $x_s(t)$ denoting the position of the shock, ξ_s its velocity and $\frac{\mu_s}{t}$ its amplitude. The function $g(\xi_s, \mu_s)$ which specifies $\mathcal{O}_g$ will be called the form factor of the operator.

By using the velocity profile (10) inside and around the shocks, one may map fields defined in terms of the velocity $u(x, t)$ into the shock representation. The two basic examples described in [10] are the generating functional $(\partial_x - \frac{\lambda}{t})\mathrm{e}^{\lambda u(x,t)}$ and $\epsilon(x, t)\mathrm{e}^{\lambda u(x,t)}$ with $\epsilon(x, t)$ the dissipation field. These fields are localized on the shocks. Indeed, outside shocks

† But no nastier singularities.

$(\partial_x u(x,t) - \frac{1}{t})$ vanishes since away from shocks, $u(x,t) = \frac{x-y_*}{t}$ with y_* almost independent of x. Similarly, the dissipation field $\epsilon(x)$, which is naively zero due to the prefactor ν in its definition, is actually a non-trivial field since $(\partial_x u)^2$ is singular in the inviscid limit. These singularities are localized on shocks and so is the dissipation field. In other words, dissipation takes place only at shocks.

The shock velocity profiles (10) at finite ν can be used to regularize the ill-defined expressions that arise in a naive $\nu = 0$ limit. In practice, given a local functional of the velocity, which is well defined at finite viscosity, one takes the $\nu = 0$ limit in the distributional sense. For the above two examples, one obtains [10]:

$$\left(\partial_x - \frac{\lambda}{t}\right) \mathrm{e}^{\lambda u(x,t)} = -2 \sum_s \mathrm{e}^{\lambda \xi_s} \sinh\left(\frac{\lambda \mu_s}{2t}\right) \delta(x - x_s(t)) \tag{24}$$

and

$$\epsilon(x,t)\mathrm{e}^{\lambda u(x,t)} = 2\lambda^{-3} \sum_s \mathrm{e}^{\lambda \xi_s} \left(\frac{\lambda \mu_s}{2t} \cosh\left(\frac{\lambda \mu_s}{2t}\right) - \sinh\left(\frac{\lambda \mu_s}{2t}\right)\right) \delta(x - x_s(t)). \tag{25}$$

Now one may use the representation of the dissipation field as an operator localized on shocks to find alternative representations of them. Indeed equation (24) implies

$$\begin{aligned}\left(u\left(\partial_x u - \frac{1}{t}\right)\mathrm{e}^{\lambda u}\right)_{(x,t)} &= -\frac{2}{\lambda^2} \sum_s \mathrm{e}^{\lambda \xi_s} \left(\lambda \xi_s \sinh\left(\frac{\lambda \mu_s}{2t}\right) + \frac{\lambda \mu_s}{2t} \cosh\left(\frac{\lambda \mu_s}{2t}\right) - \sinh\left(\frac{\lambda \mu_s}{2t}\right)\right) \\ &\quad \times \delta(x - x_s(t)).\end{aligned}$$

However, looking at the product of the operator (24) with velocity at nearby points gives

$$\frac{1}{2}\left(u(x^+,t) + u(x^-,t)\right)\left(\left(\partial_x u - \frac{1}{t}\right)\mathrm{e}^{\lambda u}\right)_{(x,t)} = -\frac{2}{\lambda^2} \sum_s \mathrm{e}^{\lambda \xi_s} \lambda \xi_s \sinh\left(\frac{\lambda \mu_s}{2t}\right) \delta(x - x_s(t)). \tag{26}$$

This is found using the fact that the velocity on the two sides of the shocks are $u_s^\pm = \xi_s \mp \frac{\mu_s}{2t}$. Comparing these expressions with the form factor of the dissipation fields, equation (25), gives

$$\lambda \epsilon(x,t)\mathrm{e}^{\lambda u(x,t)} \cong \left(\tfrac{1}{2}(u(x^+,t) + u(x^-,t)) - u(x,t)\right)(\partial_x u(x,t))\mathrm{e}^{\lambda u(x,t)}. \tag{27}$$

This is an extension of the well known formula $\epsilon(x) = \frac{1}{12} \lim_{l \to 0} \partial_l [u(x) - u(x+l)]^3$. In this formula, $u \partial_x u \mathrm{e}^{\lambda u}$ means $\partial_\lambda \frac{1}{\lambda} \partial_x \mathrm{e}^{\lambda u}$. As expected, the dissipation field is located on the discontinuity of the velocity field. This relation is valid inside any correlation functions with other fields away from point x.

Comparison of equation (20) with (27) yields an alternative way of writing the inviscid Burgers equation in which the dissipation has completely disappeared:

$$\left[\partial_t u(x,t) + \tfrac{1}{2}(u(x^+,t) + u(x^-,t))(\partial_x u(x,t))\right]\mathrm{e}^{\lambda u(x,t)} \cong 0. \tag{28}$$

This is equivalent to equation (19). It has a simple interpretation: it is the simplest possible point splitting regularization of the naive inviscid Burgers equation. The validity of this formula can be checked by hand in simple explicit correlation functions.

• Correlation functions of the velocity fields, without any derivative, are continuous as functions of the positions of the velocities. But the non-smoothness of the velocities in the inviscid limit implies that correlation functions of derivatives of the velocity field may be discontinuous and/or singular when points coincide.

This has echoes of the products of operators localized on shocks:

(i) Products of an operator localized on shocks times powers of the velocity field are discontinuous at coinciding points. These properties were illustrated in equation (26).

(ii) Products of operators localized on shocks are singular at coinciding points. More precisely, fields localized on shocks form a closed algebra [10]:

$$\mathcal{O}_f(x,t)\cdot\mathcal{O}_g(y,t) = \delta(x-y)\mathcal{O}_{fg}(x,t) + \text{regular}. \tag{29}$$

The contact term $\delta(x-y)$ in this operator product expansion arises from the coinciding shocks in the double sum representing the product operator. This operator product expansion implicitly assumes that shocks are diluted.

5. Lagrangian trajectory statistics.

For non-time-reversal invariant velocity fields one may consider backward and forward Lagrangian statistics.

• The backward statistics encodes the probability distribution of the initial positions of the trajectories at time t_0 knowing their positions at later time $t > t_0$. For n-trajectories they are given by the expectation values,

$$P^{[n]}_{ret.}(x_j,t|x^0_j,t_0) = \left\langle \prod_{j=1}^{n} \mathcal{P}_{ret.}(x_j,t|x^0_j,t_0)\right\rangle \qquad \text{with} \quad \mathcal{P}_{ret.}(x,t|x^0,t_0) = \delta(x^0 - \hat{x}(t_0|x,t)) \tag{30}$$

with $\hat{x}(t_0|x,t)$ the position of the trajectory at time t_0 which will be at x at later time $t > t_0$.

Although backward statistics are clearly a well defined object from a probabilistic point of view, our representation of backward statistics, involving $\hat{x}(t_0|x,t)$ may seem inappropriate because trajectories may merge with increasing time, so that in general trajectories cannot be followed for decreasing time. However, the measure of the set for which one or more of the points x_j lies exactly on a shock at time t is zero, so that the backward trajectory is defined with probability one. This is obviously true as long as the points x_j are all distinct. When two or more of them coincide, things are not so clear. However, we shall be able to check explicitly that our backward statistics are well normalized, i.e. that

$$\int \prod_j \mathrm{d}x^0_j \, P^{[n]}_{ret.}(x_j,t|x^0_j,t_0) = 1.$$

This ensures that we have not missed delta functions at coincident points.

• The forward statistics codes the probability distribution of the final positions of the trajectories at time t knowing their initial positions at a previous time $t_0 < t$. For n-trajectories they are given by the expectation values,

$$P^{[n]}_{adv.}(x_j,t|x^0_j,t_0) = \left\langle \prod_{j=1}^{n} \mathcal{P}_{adv.}(x_j,t|x^0_j,t_0)\right\rangle$$
$$\text{with} \quad \mathcal{P}_{adv.}(x,t|x^0,t_0) = \delta(x - x(t|x^0,t_0)) \tag{31}$$

with $x(t|x_0,t_0)$ the position of the trajectory at time t which was at x_0 at the initial time $t_0 < t$. Hence, $\hat{x}(t_0|x,t)$ and $x(t|x_0,t_0)$ are formally inverse functions: $x(t|\hat{x}(t_0|x,t),t_0) = x$. The forward probability distribution functions are normalized such that

$$\int \prod_j \mathrm{d}x_j \, P^{[n]}_{adv.}(x_j,t|x^0_j,t_0) = 1.$$

• To deal with functions and not distributions, it is convenient to compute expectation values of products of step functions:

$$H^{[n]}(x_j, t|x_j^0, t_0) = \left\langle \prod_{j=1}^{n} \mathcal{H}(x_j, t|x_j^0, t_0) \right\rangle \qquad \text{with} \quad \mathcal{H}(x, t|x^0, t_0) = \theta(x^0 - \hat{x}(t_0|x, t)) \tag{32}$$

with $\theta(z)$ the step function: $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$. The functions $H^{[n]}$ give the probabilities for particles at points x_j at time t to be at positions above x_j^0 at time t_0. They are such that

$$P_{ret.}^{[n]}(x_j, t|x_j^0, t_0) = \prod_j \partial_{x_j^0} H^{[n]}(x_j, t|x_j^0, t_0) \tag{33}$$

$$P_{adv.}^{[n]}(x_j, t|x_j^0, t_0) = (-)^n \prod_j \partial_{x_j} H^{[n]}(x_j, t|x_j^0, t_0). \tag{34}$$

Since $H^{[n]}(x_j, t|x_j^0, t_0)$ are expectation values of the local functional of the velocity field not involving derivatives they can be computed directly from the velocity distribution functions. These are recalled in appendix A.

Remark that $P_{ret.}^{[n]}$ will be regular at coinciding points since they do not involve derivatives of u, whereas $P_{adv.}^{[n]}$ will be singular since they involve such derivatives.

• Let us now argue that the backward statistics are related to the joint laws of the speeds $u(x_j, t)$, at least as long as the configuration is non-degenerate (no two points x_j coincide). In this case indeed, with probability one, no x_j lies on a shock, so each has a speed described by a single parabola, and then the same was true at any previous time. Hence with probability one, the particle passing at x_j at time t was at $x_j - (t - t_0)u(x_j, t)$ at time t_0 (remember that as long as they do not meet a shock, particles move at constant speed). So only a trivial change of variables is needed to go from the joint law of the initial positions x_j^0 to the joint law of the speeds $u(x_j, t)$, the relation being $u(x_j, t) = \frac{x_j - x_j^0}{t - t_0}$. This ensures that the total mass of the backward distribution for non-coincident points is unity, so that no finite probability is carried by degenerate configurations. This implies that

$$\mathcal{H}(x, t|x^0, t_0) = \theta(x^0 - x + (t - t_0)u(x, t)).$$

Hence the backward probability distribution is

$$\mathcal{P}_{ret.}(x, t|x^0, t_0) = \delta(x^0 - x + (t - t_0)u(x, t)). \tag{35}$$

It satisfies the adequate inviscid transport equation

$$[\partial_t + \tfrac{1}{2}(u(x^+, t) + u(x^-, t))\partial_x]\mathcal{P}_{ret.}(x, t|x^0, t_0) \cong 0 \tag{36}$$

with the appropriate boundary condition

$$\mathcal{P}_{ret.}(x, t|x^0, t_0)|_{t=t_0} = \delta(x - x_0).$$

Equation (36) is valid inside correlation functions. Note that $\mathcal{P}_{ret.}$ does not satisfy the naive transport equation (3) with $\kappa = 0$, since equation (21) yields

$$[\partial_t + u(x, t)\partial_x]\mathcal{P}_{ret.}(x, t|x^0, t_0) \cong -(t - t_0)^2\epsilon(x)\delta''(x^0 - x + (t - t_0)u(x, t)) \neq 0$$

where the left-hand side does not vanish due to the dissipative anomaly. To prove equation (36), let us expand $\mathcal{P}_{ret.}(x, t|x^0, t_0)$ in Fourier series as

$$\mathcal{P}_{ret.}(x, t|x^0, t_0) = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kx^0}\hat{\mathcal{P}}_k(x, t) \qquad \text{with} \quad \hat{\mathcal{P}}_k(x, t) = \mathrm{e}^{-\mathrm{i}k(x-(t-t_0)u(x,t))}.$$

Plugging $\hat{\mathcal{P}}_k$ into equation (36) gives

$$[\partial_t + \tfrac{1}{2}(u(x^+) + u(x^-))\partial_x]\hat{\mathcal{P}}_k(x,t) = (\mathrm{i}k)[u(x) - \tfrac{1}{2}(u(x^+) + u(x^-))]\hat{\mathcal{P}}_k(x,t)$$
$$+(\mathrm{i}k(t-t_0))[\partial_t u(x) + \tfrac{1}{2}(u(x^+) + u(x^-))(\partial_x u(x))]\hat{\mathcal{P}}_k(x,t).$$

The first term in the right-hand side vanishes since correlation functions of the velocity field without derivative are continuous whereas the second vanishes thanks to the equation of motion (28).

- The forward probability distribution is

$$\mathcal{P}_{adv.}(x,t|x^0,t_0) = -\partial_x \mathcal{H}(x,t|x^0,t_0)$$
$$= (1-(t-t_0)\partial_x u(x,t))\delta(x^0 - x + (t-t_0)u(x,t)). \qquad (37)$$

It satisfies the transport equation

$$[\partial_t + \partial_x \tfrac{1}{2}(u(x^+,t) + u(x^-,t))]\mathcal{P}_{adv.}(x,t|x^0,t_0) \cong 0 \qquad (38)$$

which corresponds to the limit $\kappa \to 0$ of equation (6). Remark that the Jacobian $(1-(t-t_0)\partial_x u(x,t))$ is always positive since away from shocks $\partial_x u = 1/t < 1/t_0$ and that on shocks $\partial_x u$ is negative.

Equation (37) implies that the forward probability distribution may be decomposed as the sum of the backward probability distribution plus an operator which is localized on shocks. Namely,

$$\mathcal{P}_{adv.}(x,t|x^0,t_0) = \frac{t_0}{t}\mathcal{P}_{ret.}(x,t|x^0,t_0) - \mathcal{D}(x,t|x^0,t_0) \qquad (39)$$

with $\mathcal{D}(x,t|x^0,t_0) = (t-t_0)(\partial_x u - \frac{1}{t})\mathcal{P}_{ret.}(x,t|x^0,t_0)$ whose shock representation is

$$\mathcal{D}(x,t|x^0,t_0) = \sum_s \chi\left(\frac{\mu_s}{2t} \geqslant |\xi_s - v_{x,x_0}|\right)\delta(x - x_s(t)) \qquad \text{with} \quad v_{x,x_0} = \frac{x - x_0}{t - t_0}$$

with $\chi(\mathcal{C})$ the characteristic function of the constraint $\mathcal{C}$. Here the constraint may also be written as $u_s^- \leqslant v_{x,x_0} \leqslant u_s^+$ which means that the speed of the trajectory going straight from (x_0,t_0) to (x,t) is between the two extreme values of the velocity at the shock.

6. Lagrangian trajectory distribution functions

The purpose of this section is to derive explicit formulae for the advanced and retarded one- and two-point function distributions of Lagrangian trajectories. We use these results to compute the short distance behaviour of these correlation functions, the probability that a particle meets a shock or that two particles get glued together. We conclude with remarks on persistence problems.

6.1. One-point functions

The one-point probability law of the velocity field $u \equiv u(x,t)$ is

$$\sqrt{\frac{t}{2\pi}}\exp\left[-\frac{tu^2}{2}\right]\mathrm{d}u.$$

This is well known since Kida [8], but rederived for completeness in appendix A. Thus the one-point PDFs for backward and forward trajectories coincide and are equal to:

$$P^{[1]}_{ret.}(x,t|x^0,t_0) = P^{[1]}_{adv.}(x,t|x^0,t_0) = \sqrt{\frac{t}{2\pi(t-t_0)^2}}\exp\left[-\frac{t(x-x^0)^2}{2(t-t_0)^2}\right]. \qquad (40)$$

It simply reflects the diffusion of the trajectories with $\langle (x-x^0)^2\rangle \simeq (t-t_0)^2/t$. For large t/t_0, this is just the ordinary dispersion of Brownian motion. But when $t-t_0$ is small compared with t_0, the dispersion grows linearly with time because with high probability no shock has been met.

It is instructive to compare this with the probability distribution for a particle starting at x_0 at time t_0 to flow to x at time t without hitting any shock. As computed in appendix B, this is equal to

$$P_{\substack{\text{no}\\ \text{shock}}}(x,t|x_0,t_0)\,\mathrm{d}x = \left(\frac{t_0}{t}\right)\sqrt{\frac{t}{2\pi(t-t_0)^2}}\exp\left[-\frac{t(x-x_0)^2}{2(t-t_0)^2}\right]\mathrm{d}x. \qquad (41)$$

In particular, the probability that a particle does not meet a shock between t_0 and t is t_0/t.

The probability distribution for a particle starting at x_0 at time t_0 to flow to x at time t hitting exactly n shocks is more complicated for $n \geqslant 0$, and it is funny that the resummations for all values of n lead to such a simple result.

6.2. Two-point functions

The two-point trajectory PDFs are slightly more lengthy to compute. The two-point PDFs for the velocity field $u_1 \equiv u(x_1,t)$ and $u_2 \equiv u(x_2,t)$ are recalled in appendix A. In the following, $F_t(z)$ stands for a variant of the error function defined by

$$F_t(z) = \mathrm{e}^{\frac{z^2}{2t}}\int_{-\infty}^{z}\mathrm{e}^{-\frac{u^2}{2t}}\,\mathrm{d}u.$$

• Let us first look at the backward probability distribution. Recall that it may be computed by a simple change of variables from the velocity distribution function. Thus for $x_1 > x_2$:

$$\begin{aligned}P^{[2]}_{ret.}(x,t|x^0,t_0) &= \frac{t^2}{(t-t_0)^2}\delta(\Delta - t(v_1-v_2))\frac{1}{F_t(-tv_2)+F_t(tv_1)}\\ &\quad+\frac{\Delta t}{(t-t_0)^2}\theta(\Delta - t(v_1-v_2))\int_{tv_1-\frac{\Delta}{2}}^{tv_2+\frac{\Delta}{2}}\mathrm{d}z\frac{\mathrm{e}^{-\frac{t}{2}(v_1^2+v_2^2)}\mathrm{e}^{\frac{z^2}{t}+\frac{\Delta^2}{4t}}}{[F_t(\frac{\Delta}{2}+z)+F_t(\frac{\Delta}{2}-z)]^2}.\end{aligned} \qquad (42)$$

Note that as expected $P^{[2]}_{ret.}$ vanishes for $t(x_1^0-x_2^0) > t_0(x_1-x_2)$ for $(x_1-x_2)>0$. (See the comments below.) Note also that in the coinciding limit $x_1 = x_2$ one has

$$\begin{aligned}P^{[2]}_{ret.}(x,t|x^0,t_0)|_{x_1=x_2} &= \delta(x_1^0-x_2^0)\sqrt{\frac{t}{2\pi(t-t_0)^2}}\exp\left[-\frac{t(x-x^0)^2}{2(t-t_0)^2}\right]\\ &= \delta(x_1^0-x_2^0)P^{[1]}_{ret.}(x,t|x^0,t_0).\end{aligned}$$

This means that two trajectories at identical final positions did start at identical initial points. The same applies to the n-trajectories probability distribution functions:

$$P^{[n]}_{ret.}(x_j,t|x_j^0,t_0)|_{x_n=x_{n-1}} = \delta(x_n^0-x_{n-1}^0)P^{[n-1]}_{ret.}(x_j,t|x_j^0,t_0). \qquad (43)$$

In other words, Lagrangian trajectories are statistically well-defined backwards.

• Consider now the forward probability distribution $P^{[2]}_{adv.}$. It is less straightforward to compute, but the relevant information can be extracted from the formula for $H^{[2]}(x_j,t|x_j^0,t_0)$, which is a sum of two contributions:

$$H^{[2]}(x_j,t|x_j^0,t_0) = K_1(x_j,t|x_j^0,t_0) + K_2(x_j,t|x_j^0,t_0).$$

Since $H^{[2]}$ is symmetric, it is enough to evaluate it for $x_1 > x_2$. To simplify the notations, we set

$$\Delta = x_1 - x_2 > 0 \qquad \text{and} \qquad v_j \equiv v_{x_j, x_j^0} = \frac{x_j - x_j^0}{t - t_0}.$$

Then

$$K_1(x_j, t|x_j^0, t_0) = \int_{\max(tv_1 - \frac{\Delta}{2}, tv_2 + \frac{\Delta}{2})}^{\infty} \frac{\mathrm{d}z}{F_t(\frac{\Delta}{2} + z) + F_t(\frac{\Delta}{2} - z)} \tag{44}$$

and

$$K_2(x_j, t|x_j^0, t_0) = \frac{\Delta}{t} \int_{tv_1}^{\infty} \mathrm{d}z_1 \int_{tv_2}^{\infty} \mathrm{d}z_2\, \theta(\Delta - (z_1 - z_2)) \\ \times \int_{z_1 - \frac{\Delta}{2}}^{z_2 + \frac{\Delta}{2}} \mathrm{d}z\, \frac{\mathrm{e}^{-\frac{1}{2t}(z_1^2 + z_2^2)} \mathrm{e}^{\frac{z^2}{t} + \frac{\Delta^2}{4t}}}{[F_t(\frac{\Delta}{2} + z) + F_t(\frac{\Delta}{2} - z)]^2}. \tag{45}$$

Of course, one could recover the results for the backward probabilities using equation (33). The explicit use of equation (34) leads to formulae for the forward probability which are not really illuminating. However, $H^{[2]}$ can be interpreted as the probability that two particles starting at time t_0 at points x_1^0 and x_2^0, respectively, have abscissae at t larger than x_1 and x_2, respectively. And indeed, one can check explicitly on the above formula for $H^{[2]}$ many expected physical properties of trajectories:

(i) Particles do not cross each other: if $\frac{x_1^0 - x_2^0}{x_1 - x_2} \leqslant \frac{t_0}{t}$ (and in particular if $(x_1^0 - x_2^0)(x_1 - x_2) \leqslant 0$), $H^{[2]}$ reduces to a one-particle distribution:

$$H^{[2]}(x_j, t|x_j^0, t_0) = \sqrt{\frac{t}{2\pi}} \int_{\max(v_1, v_2)}^{\infty} \mathrm{d}u\, \mathrm{e}^{-u^2 t/2} \\ = \begin{cases} H^{[1]}(x_1, t|x_1^0, t_0) & \text{for} \quad x_1 \geqslant x_2 \\ H^{[1]}(x_2, t|x_2^0, t_0) & \text{for} \quad x_1 \leqslant x_2. \end{cases} \tag{46}$$

Taking derivatives with respect to x_1 and x_2, one finds a vanishing probability density if the respective orders of the particle positions have changed between initial and final times.

(ii) Trajectories are well-defined forward: for fixed x_1,x_2 and t, formula (46) is valid for $|x_1^0 - x_2^0|$ small enough, and leads to

$$\lim_{x_1^0, x_2^0 \to x^0} H^{[2]}(x_1, x_2, t|x_1^0, x_2^0, t_0) = H^{[1]}(\max(x_1, x_2), t|x^0, t_0) \\ = \sqrt{\frac{t}{2\pi}} \int_{\frac{\max(x_1, x_2) - x^0}{t - t_0}}^{\infty} \mathrm{d}u\, \mathrm{e}^{-u^2 t/2}.$$

Taking the derivatives with respect to x_1 and x_2 gives

$$\lim_{x_1^0, x_2^0 \to x^0} P^{[2]}_{adv.}(x_1, x_2, t|x_1^0, x_2^0, t_0) = \delta(x_1 - x_2) P^{[1]}_{adv.}(x, t|x^0, t_0) \\ = \delta(x_1 - x_2) \sqrt{\frac{t}{2\pi(t - t_0)^2}} \exp\left[-\frac{t(x - x^0)^2}{2(t - t_0)^2}\right]. \tag{47}$$

• Contrary to the backward PDF, $P^{[2]}_{adv.}$ is singular at coinciding points: assuming $x_1^0 \neq x_2^0$ a direct computation shows that

$$P^{[2]}_{adv.}(x_j, t|x_j^0, t_0) = R(x_1, t|x_j^0, t_0)\delta(x_1 - x_2) + \cdots. \tag{48}$$

The dots refer to terms regular at $x_1 = x_2$. The coefficient $R(x, t|x_j^0, t_0)$, which has the dimension of the inverse of a length, is the probability density of aggregation of trajectories at point x. It is equal to

$$R(x, t|x_1^0, x_2^0, t_0) = \frac{\mathrm{e}^{-\frac{t}{2}(v_1^2+v_2^2)}}{2\pi t}[F_t(tv_1) + F_t(-tv_2)] \qquad \text{for} \quad v_1 \leqslant v_2. \quad (49)$$

Let us note that this also gives the probability that n particles have collapsed, if v_1 and v_2 refer to the speeds of the particles with the extreme initial positions.

This formula simplifies if one is simply interested in the probability that two particles starting at distinct points at t_0 have glued together at time t: integration over the final position gives for the total gluing probability

$$\frac{t - t_0}{t} \int_{\frac{|x_1^0 - x_2^0|\sqrt{t}}{t-t_0}}^{\infty} \frac{\mathrm{d}v}{\sqrt{\pi}} \mathrm{e}^{-v^2/4}. \qquad (50)$$

For fixed t_0 and $x_1^0 - x_2^0$, and $t \to \infty$, this behaves like $1 - |x_1^0 - x_2^0|/\sqrt{\pi t} - t_0/t$. This shows that distinct particles are sure to be at the same point at a late enough moment of the evolution, but this gluing occurs rather slowly.

Another special case where the general formula simplifies is the limit of identical initial positions $x_1^0 = x_2^0 = x^0$ leading to

$$R|_{x_1^0 = x_2^0} = \sqrt{\frac{1}{2\pi t}} \exp\left[-\frac{t(x - x^0)^2}{2(t - t_0)^2}\right].$$

This should be compared with formula (47). The difference is exactly equal to the probability to go without shock from x^0 to x in the time interval $[t_0, t]$ (41), and this has a good explanation: two particles starting at the same point stay stuck together, two particles starting at distinct point may coalesce only when they meet a shock, so the difference in collapse between starting at the same point and starting infinitely close is simply encoded in the probability that a single particle has met no shock. Integration over the final points shows that the probability for two infinitely close particles at time t_0 to have glued together at time t is $1 - t_0/t$.

The collapse probability R may be computed in other ways. One way consists in using the operator product algebra of operator localized on shocks, cf equation (29). Indeed, in view of the decomposition (39) of $\mathcal{P}_{adv.}(x, t|x^0, t_0)$, one has the following operator product expansion:

$$\begin{aligned}\mathcal{P}_{adv.}(x_1|x_1^0)\mathcal{P}_{adv.}(x_2|x_2^0) &= \mathcal{D}(x_1|x_1^0)\mathcal{D}(x_2|x_2^0) + \text{regular} \\ &= \mathcal{R}(x_1|x_j^0)\delta(x_1 - x_2) + \cdots\end{aligned}$$

with $\mathcal{R}$ the operator localized on shocks whose form factor is the product of those of the operator $\mathcal{D}(x|x^0)$, i.e.

$$\mathcal{R}(x|x_j^0) = \sum_s \chi\left(\frac{\mu_s}{2t} \geqslant \max_j |\xi_s - v_j|\right)\delta(x - x_s(t)).$$

Clearly, using the shock distribution recalled in the appendix, one gets $\langle\mathcal{R}(x|x_j^0)\rangle = R(x, t|x_j^0, t_0)$ as computed in equation (49).

Another way to compute $R(x, t|x_j^0, t_0)$ is as follows. We know that particles do not cross, so that from the equation of trajectories, we can infer that two particles starting at x_1^0 and x_2^0 ($x_1^0 > x_2^0$), respectively, are glued together between x and $x + \mathrm{d}x$ at time t if and only if $x - x_2 - (t - t_0)u(x, t) \leqslant 0$ (i.e. $u(x, t) \geqslant v_2$) and $x + \mathrm{d}x - x_1 - (t - t_0)u(x + \mathrm{d}x, t) \geqslant 0$

(i.e. $u(x + dx, t) \leqslant v_1$). But as recalled in the appendix on shock distribution functions, the probability that $u(x, t) \geqslant v_2$ and $u(x + dx, t) \leqslant v_1$ for $v_2 \geqslant v_1$ is

$$\frac{dx}{2\pi} \int_{v_2}^{\infty} dv_+ \int_{-\infty}^{v_1} dv_- \, t(v_+ - v_-)\theta(v_+ - v_-) e^{-t(v_+^2+v_-^2)/2}. \tag{51}$$

This leads again to the above formula for $R(x, t|x_j^0, t_0)$.

6.3. *A comment on persistence problems*

To every random velocity distribution, one can associate domains on the x-axis, defined as the intervals where the velocity v is continuous. Those domains change as shocks move and annihilate into other shocks. This is a typical situation where persistence concepts are useful. We have computed above two quantities that relate naturally to persistence. For instance the probability to move on a Lagrangian trajectory in the time interval $[t_0, t]$ without meeting a shock, i.e. remaining in the same domain was found to be t_0/t. In the same vein, the probability for two particles starting on Lagrangian trajectories at distance $x > 0$ from each other at time t_0 to be at distinct positions at time t was found to be

$$1 - \frac{t - t_0}{t} \int_{\frac{x\sqrt{t}}{t-t_0}}^{\infty} \frac{dv}{\sqrt{\pi}} e^{-v^2/4} \tag{52}$$

which behaves for large t and fixed x and t_0 as

$$\frac{x}{\sqrt{\pi t}} + \frac{t_0}{t}. \tag{53}$$

In particular there is no unexpected persistence exponent.

Let us note that the more usual definition of persistence, which is not related to Lagrangian trajectories but deals with points that do not move with time, leads to a different kind of behaviour† that can be computed by direct use of the distribution of velocities. For instance, the probability that a fixed point (say, the origin) is not hit by any shock in the interval $[t_0, t]$ is

$$\int dy \left[\int dy' \, \exp\left(\sup_{t' \in [t_0, t]} \frac{y^2 - y'^2}{2t'} \right) \right]^{-1}. \tag{54}$$

In the limit $t/t_0 \to \infty$, this exhibits the slightly non-trivial behaviour

$$\left(\frac{2}{\pi} \left(\frac{t_0}{t} \right) \log \left(\frac{t}{t_0} \right) \right)^{1/2} \tag{55}$$

quite different from the previous results for moving particles.

7. Inverse cascade

We now consider properties of a tracer advected in the inviscid Burgers decaying turbulence. In particular, we argue that there is no dissipative anomaly and that the energy cascade is inverse.

• As previously explained, in the inviscid limit the appropriate transport equations are equations (5), (6). In the limit $\kappa \to 0$, their solutions may be written in terms of the backward and forward probability distributions. Namely,

$$\begin{aligned} T(x, t) &= \int dx^0 \, \mathcal{P}_{ret.}(x, t|x^0, t_0) T_0(x^0) \\ &= T_0(x - (t - t_0)u(x, t)) \end{aligned} \tag{56}$$

† This comparison was suggested to us by Claude Godrèche.

and

$$\rho(x,t) = \int \mathrm{d}x^0\, \mathcal{P}_{adv.}(x,t|x^0,t_0)\rho_0(x^0)$$
$$= (1-(t-t_0)\partial_x u(x,t))\rho_0(x-(t-t_0)u(x,t)) \tag{57}$$

where $T_0(x^0)$ and $\rho_0(x^0)$ are the initial conditions at time t_0.

Since correlations of the trajectory probability distributions are computable, it is not hard to evaluate correlations of the scalars. Let us illustrate this by showing that there is no dissipation of energy for the tracer $T(x,t)$ and hence no dissipation anomaly for T. The mechanism for that property is similar to the one described in [5] in the case of the compressible Kraichnan's model. Assume that one is given the translation invariant two-point function of the initial data:

$$\langle T_0(x_1)T_0(x_2)\rangle = \Gamma(x_1-x_2).$$

The density of energy of the tracer is $\mathcal{E}(x,t) = \frac{1}{2}T^2(x,t)$. Its average is

$$\langle \mathcal{E}(t)\rangle = \tfrac{1}{2}\int \mathrm{d}x_1^0\,\mathrm{d}x_2^0\, P^{[2]}_{ret.}(x,x|x_1^0,x_2^0)\langle T_0(x_1^0)T_0(x_2^0)\rangle$$
$$= \tfrac{1}{2}\int \mathrm{d}x^0\, P^{[1]}_{ret.}(x|x^0)\Gamma(0) = \tfrac{1}{2}\Gamma(0)$$

where we have used equation (43) for $P^{[2]}_{ret.}$ at coinciding points and the normalization condition for $P^{[1]}_{ret.}$. Thus energy is conserved in mean, $\langle\mathcal{E}(t)\rangle = \overline{\mathcal{E}_0}$, and this is due to the fact that the trajectories are statistically well-defined backward. Notice, however, that at fixed initial data the density of energy decreases at large time as $\langle\mathcal{E}(x,t)\rangle \simeq \frac{1}{\sqrt{2\pi t}}\int \mathrm{d}y\, T_0^2(y)$ if the integral converges.

More generally, the well-defined character of the trajectories may also be formulated as the following operator product identity:

$$\mathcal{P}_{ret.}(x,t|x_1^0,t_0)\mathcal{P}_{ret.}(x,t|x_1^0,t_0) = \delta(x_1^0-x_2^0)\mathcal{P}_{ret.}(x,t|x_1^0,t_0).$$

As a consequence, any products of solutions of the transport equation (5) at $\kappa = 0$ will also be solution. In particular, any powers of $T(x,t)$ are also solutions:

$$\partial_t T^n(x,t) + \tfrac{1}{2}(u(x^+,t)+u(x^-,t))\partial_x T^n(x,t) \cong 0$$

inside correlation functions. This shows the absence of dissipative anomalies in the passive free advection which means that the fields $\kappa T^n\partial_x^2 T$ vanish inside correlation functions at $\kappa = 0$.

• This is the sign of the absence of a direct energy cascade, as in the two-dimensional turbulence in which the energy cascade is inverse, i.e. toward the large scales [11]. To show it more explicitly let us now assume that one is injecting energy to the tracer such that the transport equation is now

$$\partial_t T(x,t) + \tfrac{1}{2}(u(x^+,t)+u(x^-,t))\partial_x T(x,t) = f(x,t) \tag{58}$$

with $f(x,t)$ the forcing term. Solutions of this equation with zero initial data at time t_0 are:

$$T(x,t) = \int_{t_0}^{t}\mathrm{d}s\int \mathrm{d}y\, \mathcal{P}_{ret.}(x,t|y,s)f(y,s)$$
$$= \int_{t_0}^{t}\mathrm{d}s\, f(x-(t-s)u(x,t),s). \tag{59}$$

Assume that the two-point function of the force is delta-correlated in time:

$$\langle f(y_1,s_1)f(y_2,s_2)\rangle = C_L(y_1-y_2)\delta(s_1-s_2) \tag{60}$$

with $C_L(x)$ a smooth function varying on scale L and with rapid decrease at infinity. The energy injection rate is $\bar{e} = \frac{1}{2}C_L(0)$. Using again the fact that trajectories are well-defined backward, equation (43), one finds that the average of the tracer energy density at time t is:

$$\langle \mathcal{E}(t) \rangle = \frac{1}{2} \int_{t_0}^{t} \mathrm{d}s \int \mathrm{d}y \, P^{[1]}_{ret.}(x, t|y, s) \, C_L(0)$$
$$= \frac{1}{2}(t - t_0)C_L(0) = (t - t_0)\bar{e} \tag{61}$$

where, again, we used the well definedness of the trajectories (see equation (43)) and the normalization of the probabilities. Thus, the total amount of energy injected into the system is transfered without dissipation.

To decipher in which mode the energy is injected, let us consider the scalar two-point function at distinct points. For forcing delta-correlated in time as in equation (60), the two-point function is

$$\langle T(x_2, t)T(x_1, t) \rangle = \int_{t_0}^{t} \mathrm{d}s \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, P^{[2]}_{ret.}(x_j, t|y_j, s) C_L(y_1 - y_2).$$

It behaves at large time and fixed positions as

$$\langle T(x_2, t)T(x_1, t) \rangle = (t\sqrt{\pi} - |x_2 - x_1|\sqrt{t}) \int_0^1 \frac{\mathrm{d}s}{\sqrt{\pi}} C_L(s|x_2 - x_1|) + F(x_2, x_1) + \mathrm{O}(1/\sqrt{t})$$

with $F(x_2, x_1)$ finite as $t \to \infty$ scaling as $|x_2 - x_1|$ at small distance. The energy is thus transfered to the mode corresponding to the first line of the above equation. Its amplitude increases with time. It is a soft, although non-constant, mode varying smoothly and slowly.

To make manifest the absence of dissipation, consider products of the forced scalar (59) at coincident points. One has

$$T^n(x, t) = \prod_{j=1}^{n} \int_{t_0}^{t} \mathrm{d}s_j \, f(x - (t - s_j)u(x, t), s_j).$$

Using again equation (36) or (28), one deduces that inside correlation functions

$$\partial_t T^n(x, t) + \frac{1}{2}(u(x^+, t) + u(x^-, t))\partial_x T^n(x, t) \cong nf(x, t)T^{n-1}(x, t).$$

This shows that there is no dissipative anomalies at $\kappa = 0$ in the scalar advection. Note that what we have described is a limit when κ goes to zero first and then t goes to infinity to reach the stationary state.

Again, the mechanism is similar to the one found in compressible Kraichnan's models [5]: the injected energy is accumulated in the soft mode, there is no dissipative anomaly and the energy cascade is inverse. This is directly related to the fact that the trajectories are statistically well defined.

Acknowledgments

We thank K Gawedzki and M Vergassola for communicating us their results [5] prior to publication.

Appendix A. Velocity probability distributions

In this appendix, we recall known formulae for the one- and two-point probability distributions for velocities (see e.g. [8]). We just give a reminder of the computational rules and illustrate it in the case of the one-point-velocity PDF. A further illustration is given in appendix B.

We define $S(x,t) = \min_j(\phi_j + \frac{(x-y_j)^2}{2t})$ so that $u(x,t) = \partial_x S(x,t)$. The pairs (ϕ_j, y_j) are described by a Poisson point process, saying that the cell of size $\mathrm{d}\phi\,\mathrm{d}y$ in the (ϕ, y)-plane is occupied with probability $\mathrm{e}^{\phi}\,\mathrm{d}\phi\,\mathrm{d}y$, disjoint cells being independent. This leads to the following useful fact that if D is any measurable set in the (ϕ, y)-plane, the probability that all cells in D are empty is $\mathrm{e}^{-\int_D \mathrm{e}^{\phi}\,\mathrm{d}\phi\,\mathrm{d}y}$. We call that pair (ϕ_j, y_j) giving the minimum of S at the point (x,t) the parameters at (x,t).

A.1. One-point-velocity PDF

We look for the probability $P(u(x,t) \in [v, v+\mathrm{d}v])$. The law for the Poisson point process reads in this case:

- A cell (ϕ, y) with $\frac{x-y}{t} \in [v, v+\mathrm{d}v]$ is occupied.
- The cells in $D = \{(\phi', y')$ such that $\phi' + \frac{(x-y')^2}{2t} < \phi + \frac{(x-y)^2}{2t} = \phi + \frac{v^2 t}{2}\}$ are empty.

  Therefore,

$$P(u(x,t) \in [v, v+\mathrm{d}v]) = \int_{\frac{x-y}{t}\in[v,v+\mathrm{d}v]} \mathrm{e}^{\phi}\,\mathrm{d}\phi\,\mathrm{d}y \mathrm{e}^{-\int_D \mathrm{e}^{\phi'}\,\mathrm{d}\phi'\,\mathrm{d}y'}.$$

Let us perform this computation in detail. First we put the integral over ϕ', which varies between $-\infty$ and $\phi + \frac{(x-y)^2}{2t} - \frac{(x-y')^2}{2t}$. This yields

$$P(u(x,t) \in [v, v+\mathrm{d}v]) = \int_{\frac{x-y}{t}\in[v,v+\mathrm{d}v]} \mathrm{e}^{\phi}\,\mathrm{d}\phi\,\mathrm{d}y\, \mathrm{e}^{-\int \mathrm{e}^{\phi+(x-y)^2/2t-(x-y')^2/2t}\,\mathrm{d}y'}.$$

Then, we integrate over ϕ to get

$$P(u(x,t) \in [v, v+\mathrm{d}v]) = \int_{\frac{x-y}{t}\in[v,v+\mathrm{d}v]} \mathrm{d}y\, \frac{\mathrm{e}^{-(x-y)^2/2t}}{\int \mathrm{e}^{-(x-y')^2/2t}\,\mathrm{d}y'}.$$

Let us note that the possibility to integrate explicitly over the variable ϕ' parametrizing the empty domain D and over the 'centre of mass' of the variables ϕ parametrizing the occupied cells is typical. In this explicit example, the other integrations are also immediate, but this is rather unusual.

The y' integral gives a factor $1/\sqrt{2\pi t}$ and the integration domain for y is infinitesimal, so $y = x - vt$ and $\mathrm{d}y = t\,\mathrm{d}v$. Finally:

$$P(u(x,t) \in [v, v+\mathrm{d}v]) = \sqrt{\frac{t}{2\pi}}\mathrm{e}^{-tv^2/2}\,\mathrm{d}v.$$

Let us observe that it has total mass 1, ensuring that this computation, which does not take shocks into account, does not miss any event of non-zero measure. This is a sign that the shocks are diluted.

A.2. Two-point-velocity PDF

We look for the probability $P(u(x_1,t) \in [v_1, v_1+\mathrm{d}v_1], u(x_2,t) \in [v_2, v_2+\mathrm{d}v_2])$. By symmetry, we may (and shall) assume $x_1 - x_2 \equiv \Delta > 0$. There are two possibilities:

(i) One parabola.

- A cell (ϕ, y) with $\frac{x_1-y}{t} \in [v_1, v_1+\mathrm{d}v_1]$ and $\frac{x_2-y}{t} \in [v_2, v_2+\mathrm{d}v_2]$ is occupied.
- The cells in $D = \{(\phi', y')$ such that $\phi' + \frac{(x_1-y')^2}{2t} < \phi + \frac{(x_1-y)^2}{2t}$ or $\phi' + \frac{(x_2-y')^2}{2t} < \phi + \frac{(x_2-y)^2}{2t}\}$ are empty.

(ii) Two parabolae.

- A cell (ϕ_1, y_1) with $\frac{x_1-y_1}{t} \in [v_1, v_1+\mathrm{d}v_1]$ and a cell (ϕ_2, y_2) with $\frac{x_2-y_2}{t} \in [v_2, v_2+\mathrm{d}v_2]$ are occupied, such that $\phi_1 + \frac{(x_1-y_1)^2}{2t} < \phi_2 + \frac{(x_1-y_2)^2}{2t}$ and $\phi_2 + \frac{(x_2-y_2)^2}{2t} < \phi_1 + \frac{(x_2-y_1)^2}{2t}$.
- The cells in $D = \{(\phi', y')$ such that $\phi' + \frac{(x_1-y')^2}{2t} < \phi_1 + \frac{(x_1-y_1)^2}{2t}$ or $\phi' + \frac{(x_2-y')^2}{2t} < \phi_2 + \frac{(x_2-y_2)^2}{2t}\}$ are empty.

Accordingly, $P(u(x_1, t) \in [v_1, v_1 + \mathrm{d}v_1], u(x_2, t) \in [v_2, v_2 + \mathrm{d}v_2])$ is a sum of two contributions $P_{1\text{parab}}$ and $P_{2\text{parab}}$ which are found after some computation to be:

$$P_{1\text{parab}} = t^2\, \mathrm{d}v_1\, \mathrm{d}v_2\, \delta(\Delta - t(v_1 - v_2)) \frac{1}{F_t(-v_2 t) + F_t(v_1 t)}$$

and

$$P_{2\text{parab}} = \Delta t\, \mathrm{d}v_1\, \mathrm{d}v_2\, \theta(\Delta - t(v_1 - v_2)) \mathrm{e}^{-t(v_1^2+v_2^2)/2+\Delta^2/4t} \times \int_{tv_1-\Delta/2}^{tv_2+\Delta/2} \mathrm{d}z\, \frac{\mathrm{e}^{z^2/t}}{[F_t(\frac{\Delta}{2} - z) + F_t(\frac{\Delta}{2} + z)]^2}.$$

Let us recall that $F(x) \equiv \frac{\mathrm{e}^{x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{d}y\, \mathrm{e}^{-y^2/2}$ and $F_t(x) \equiv \sqrt{2\pi t}\, F(x/\sqrt{t})$.

Again, one can check explicitly that the sum has total mass 1, or even better that the integral over v_1 or v_2 gives again the one-point PDF. This computation shows that $P_{1\text{parab}}$ which lives on a codimension-one hyperplane, is completely determined as a kind of boundary of $P_{2\text{parab}}$.

A.3. Distribution of shocks

The two-point PDF for velocities allows us to compute the probability to have a shock such that $u(x, t) = v_+$ and $u(x + \mathrm{d}x, t) = v_-$ between x and $x + \mathrm{d}x$ by taking $\Delta \to 0$. The result is

$$\frac{\mathrm{d}x}{2\pi}\, \mathrm{d}v_+\, \mathrm{d}v_-\, t(v_+ - v_-)\theta(v_+ - v_-)\mathrm{e}^{-t(v_+^2+v_-^2)/2}. \tag{62}$$

This can be expressed as the probability of finding a shock of amplitude $\mu/t = v_+ - v_-$ and velocity $\xi = (v_+ + v_-)/2$ in the interval $[x, x + \mathrm{d}x]$ as

$$\frac{\mathrm{d}x}{2\pi t}\, \mathrm{d}\mu\, \mathrm{d}\xi\, \mu\theta(\mu)\mathrm{e}^{-\xi^2 t-\mu^2/4t}. \tag{63}$$

In particular, the probability to have a shock in the interval $[x, x + \mathrm{d}x]$ is $\mathrm{d}x/\sqrt{\pi t}$. This involves only configurations with two parabolae, whereas the probability that there is no shock in a finite interval $[x, x']$ is computed with configurations involving one parabola and found to be $\int \mathrm{d}y\, (F_t(y - x) + F_t(x' - y))^{-1}$. This makes it intuitively (if not mathematically) clear that with probability one a finite interval contains only a finite number of shocks.

Appendix B. One-point PDF without shock

In this appendix, we compute the probability $P(x, t|x_0, t_0)_{\text{no shock}}$ that a particle starting at point x_0 at time t_0 arrives in $[x, x + \mathrm{d}x]$ at time t without ever meeting a shock. This corresponds to the following configuration:

- A cell (ϕ, y) with $x_0 + \frac{x_0-y}{t_0}(t - t_0) \in [x, x + \mathrm{d}x]$ is occupied. Let $v = (x_0 - y)/t_0$.
- The cells in $D = \{(\phi', y')$ such that $\phi' + \frac{(x_0+v(t'-t_0)-y')^2}{2t'} < \phi + \frac{(x_0+v(t'-t_0)-y)^2}{2t'}$ for some $t' \in [t_0, t]\}$ are empty.

The second constraint seems complicated. We claim that it is equivalent to the extreme constraint for $t' = t$:

- The cells in $D = \{(\phi', y')$ such that $\phi' + \frac{(x_0+v(t-t_0)-y')^2}{2t} < \phi + \frac{(x_0+v(t-t_0)-y)^2}{2t}\}$ are empty.

This is a direct consequence of an important property of trajectories. As already stated before, Lagrangian trajectories stick to shocks as soon as they meet one. We can even be a bit more precise. Suppose that at time (x, t) the parabola of parameters (ϕ, y) dominates (ϕ', y'), so

$$\phi + \frac{(x-y)^2}{2t} < \phi' + \frac{(x-y')^2}{2t} \tag{64}$$

or better

$$\phi - \phi' < \frac{(y-y')(2x-y-y')}{2t}. \tag{65}$$

Consider a fictive particle moving at constant speed $v = (x-y)/t$ and arriving at point x at time t. At time $t_0 < t$ it was at point $x_0 = x - v(t-t_0)$. The identity

$$\frac{(y-y')(2x_0-y-y')}{2t_0} - \frac{(y-y')(2x-y-y')}{2t} = \frac{t-t_0}{2tt_0}(y-y')^2 > 0 \tag{66}$$

proves that the parabola of parameters (ϕ, y) was already dominant at (x_0, t_0). This proves that the equivalence of the two above definitions of the forbidden domain D. This also means that if point x is not on a shock at time t and $u(x, t) = v$, there is a unique backward Lagrangian trajectory through (x, t), defined back to time t_0 and such that at t_0 the particle was at point $x_0 = x - v(t-t_0)$.

So we need to compute

$$\int_{x_0+\frac{x_0-y}{t_0}(t-t_0)\in[x,x+\mathrm{d}x]} \mathrm{e}^{\phi}\,\mathrm{d}\phi\,\mathrm{d}y\,\mathrm{e}^{-\int_D \mathrm{e}^{\phi'}\,\mathrm{d}\phi'\,\mathrm{d}y'}.$$

Again, integration over ϕ', ϕ and y' is straightforward, and yields

$$P_{\substack{\text{no}\\\text{shock}}}(x, t|x_0, t_0)\,\mathrm{d}x = \left(\frac{t_0}{t}\right)\sqrt{\frac{t}{2\pi(t-t_0)^2}}\exp\left[-\frac{t(x-x_0)^2}{2(t-t_0)^2}\right]\mathrm{d}x.$$

Hence, the probability that no shock is met in the interval $[t_0, t]$ is simply t_0/t.

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